AIAA JOURNAL
Vol. 46, No. 7, July 2008

# Best Strip-Beam Properties Derivable from Classical Lamination Theory

Wenbin Yu*
*Utah State University, Logan, Utah 84322-4130*
and
Dewey H. Hodges†
*Georgia Institute of Technology, Atlanta, Georgia 30332-0150*

DOI: 10.2514/1.34182

**This paper presents a simple method to extract classical beam properties for composite strips with general anisotropy from an asymptotically correct classical lamination theory derived using the variational-asymptotic method. None of the ad hoc approximations common to the literature of beam theory are used. Rather, this approach discovers the necessary and sufficient engineering assumptions for obtaining the correct cross-sectional constants and stress recovery relations for a strip. Numerical examples are provided to illustrate the resulting theory. Resulting elastic constants agree well with those from well-established methods of calculating cross-sectional constants, and the accuracy deteriorates as the strip thickness-to-width ratio increases.**

## Nomenclature

| | | |
|---|---|---|
| $A$ | = | cross-sectional area |
| $\mathbf{B}_i$ | = | base vectors of the coordinate system after deformation |
| $\mathbf{b}_i$ | = | base vectors of the coordinate system before deformation |
| $E$ | = | Young's modulus |
| $EI$ | = | bending stiffness |
| $F_{ij}$ | = | mixed-basis deformation gradient tensor |
| $\mathbf{R}$ | = | through-the-thickness average of the position vector $\hat{\mathbf{R}}$ |
| $\hat{\mathbf{R}}$ | = | position vector of a material point after deformation |
| $\mathbf{r}$ | = | position vector of a material point on the undeformed reference surface |
| $\hat{\mathbf{r}}$ | = | position vector of a material point before deformation |
| $v_i$ | = | warping functions for the cross section |
| $w_i$ | = | warping functions for the transverse normal |
| $\Gamma_{ij}$ | = | three-dimensional strain tensor |
| $\gamma_{11}$, $\kappa_i$ | = | one-dimensional generalized beam strains |
| $\varepsilon_{\alpha\beta}$, $K_{\alpha\beta}$ | = | two-dimensional generalized plate strains |
| $\nu$ | = | Poisson's ratio |

## Introduction

CLASSICAL lamination theory (CLT) of plates is suitable for the modeling of thin, isotropic plates and for extremely thin composite plates. Typical derivations of this theory start with assumptions, such as the normal line element being rigid and normal to the deformed midsurface of the plate [1]. Unless one first makes the (contradictory) assumption that the stress through the thickness of the plate is zero, this assumption leads to an overly stiff plate theory. Asymptotic methods, on the other hand, have shown that the assumption of a rigid normal line element is neither necessary nor correct. On the other hand, the assumption of zero transverse normal and shear stresses leads to expressions for the corresponding strain components, which can in fact be used to recover nonzero through-thickness displacements induced by Poisson contraction. Approached this way, the resulting theory is asymptotically exact for the limit of zero thickness. The resulting theory is thus no more complicated, but it is more general [2,3].

A similar problem has existed along side this one for decades. Typical derivations of beam theory often start with such assumptions as the cross section being rigid in its own plane. Again, if one has not *already* reduced the stress-strain law by setting equal to zero both the transverse normal stresses and the shear stress in the plane of the cross section, this approach will lead to an overly stiff beam theory. Asymptotic methods have shown that, for isotropic beams, the assumptions of zero transverse normal stresses and zero shear stress in the cross-sectional plane are sufficient. One thereby derives expressions for the corresponding strain components that may be used to recover nonzero in-plane displacements induced by Poisson contraction.

Although these observations are well known and frequently publicized, there is one corner of this spectrum of problems, however, that has not had sufficient exposure in the literature and still evidently remains a mysterious matter to some: the reduction of CLT, when applied to a strip, to formulate a strip-beam theory. The purpose of this paper is to attempt to clear up sources of this confusion.

In the seminal and foundational work of Vlasov [4] for thin-walled beams, one finds his assumptions clearly stated up front, the first of which is that the cross section remains rigid in its own plane. Unfortunately, consistent application of this assumption leads to a material coefficient $E/(1-\nu^2)$ instead of $E$ in the formulas for axial stiffness ($EA$) and bending stiffness ($EI$) of elementary beam theory; that way, these stiffnesses are overestimated by about 10% for $\nu = 0.3$. However, in his application of the constitutive relations, Vlasov did *not* follow this assumption. Instead, he first neglected stresses in the contour directions, reducing the constitutive law. This is analogous to applying the Bernoulli hypothesis (i.e., that the stress field in the beam is uniaxial) to the development of classical beam theory before assuming anything about the kinematics. Therefore, the correct axial and bending stiffnesses of classical beam theory were obtained. Following this approach, one need *not* set the in-plane distortion caused by Poisson effects equal to zero; instead, it simply does not show up in the theory. However, if desired, one can use the expressions for transverse normal strains and cross-sectional in-plane distortion shear strain that come from reducing the constitutive law to find the in-plane displacement distribution over the cross section.

Received 21 August 2007; revision received 6 February 2008; accepted for publication 13 February 2008. 

*Assistant Professor, Department of Mechanical and Aerospace Engineering. Senior Member AIAA.

†Professor, School of Aerospace Engineering. Fellow AIAA.

Because Vlasov theory is not incorrect for isotropic beams, the inconsistency regarding his first assumption would not amount to much in the grand scheme of things. However, extension of the Vlasov theory to anisotropic beams has attracted significant attention from researchers [5–8]. Such theories attempt to construct beam models based on classical, laminated plate/shell theory in conjunction with the kinematic assumptions that were originally used in Vlasov theory for isotropic beams. In particular, in some of these works the beam cross section is assumed to be rigid in its own plane before any reduction of the constitutive law, and the transverse shear strains are neglected [5,6]. As proven in detail in Volovoi et al. [9,10], such assumptions lead to contradictions and inaccuracies, even for isotropic beams, and the consequences are even less predictable for the generally anisotropic case. For example, Bert [11] simply integrated through the thickness the piecewise constant, plane stress reduced lamina moduli to obtain the beam stiffness. Vinson and Sierakowski [12] developed what they termed “advanced beam theory” from classical lamination theory by neglecting lateral strains and curvatures. These two approaches can be easily shown to lead to incorrect results by considering the special case of isotropic materials: the corresponding bending stiffness of the beam will be $EI/(1-\nu^2)$ instead of the well-known and correct value of $EI$.

There exists an alternative approach to constructing thin-walled beam theories that avoids ad hoc kinematic assumptions and relies instead on equilibrium equations. This, in general, leads to more rigorous thin-walled beam theories [13,14]. Some attempts to apply this method to the development of Vlasov theory were made [8], but the procedure is not straightforward.

Some researchers, such as Wu and Sun [7] and Gjelsvik [15], tacitly recommend the neglect of stresses, correctly noting that the alternative of neglecting strains leads to an overly stiff structural model. Others [5,6] consistently followed Vlasov’s explicitly stated assumption and neglected in-plane strains of the cross section. Although the resulting material coefficients satisfying these two contradictory assumptions differ by a factor of $1-\nu^2$ for isotropic beams, which happens to be close to unity, the difference for anisotropic beams can be dramatic for certain layups. Results presented in Chapter 6 of Hodges [16] and the references cited therein prove the validity of neglecting stresses in the contour direction (i.e., what Vlasov actually did) and the invalidity of Vlasov’s first stated assumption. The correct cross-sectional elastic constants derived in the present paper are identical to those that can be obtained by following the procedures outlined in Wu and Sun [7] and Reissner and Tsai [13] if the in-plane stress resultant and moment along the contour direction together with the membrane shear resultant are set to zero. In an analogous manner for strip beams, Barbero et al. [17] obtained beam properties based on the reciprocals of the corresponding laminated compliances originally proposed by Whitney et al. [18] along with the assumptions that laminate resultant force and moment originated by the normal stress in the width direction are zero. (Unfortunately, torsion is excluded by assuming the resultant twisting moment is also zero.) The same method is also used by Reddy to derive a laminated beam theory in his comprehensive treatment of composite structures [19]. Because all these approaches are derived in an ad hoc manner based on engineering intuition, there is currently no agreement on which set of assumptions will produce the best beam properties [20]. Furthermore, a complete set of properties for a classical beam model capable of dealing with extension, torsion, and bending simultaneously and handling all elastic couplings does not appear to be well known to engineers who might find them useful [16,21].

This paper will identify the necessary engineering assumptions to extract the best beam properties from CLT and provide simple closed-form formulas for the equivalent beam properties for the quick reference of practicing engineers. We proceed by first generalizing our previous work on variational-asymptotic modeling of composite plates [3,22–24] to develop an asymptotically correct CLT suitable for composite plates made of materials having full anisotropy. Then, we show how the variational-asymptotic method leads to a simple and rigorous extraction of equivalent beam properties for a strip when CLT is used as a starting point. The necessary and sufficient set of engineering assumptions to accomplish this same reduction is also presented. Finally, numerical results are presented.

## Asymptotically Correct Classical Lamination Theory

We introduce Cartesian coordinates $x_i$ to locate an arbitrary point in the plate, where $x_\alpha$ are two orthogonal lines in the reference surface (we choose the middle surface for convenience) and $x_3$ is the normal coordinate (see Fig. 1). (Here and throughout the paper, Greek indices assume values of 1 and 2 whereas Latin indices assume values of 1, 2, and 3. Repeated indices are summed over their range except when explicitly indicated.) Letting $\mathbf{b}_i$ denote the unit vector along $x_i$ for the undeformed plate, we can then describe the position of any material point in the undeformed configuration by its position vector $\hat{\mathbf{r}}$ from a fixed point $O$, such that

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3\mathbf{b}_3 \tag{1}$$

where $\mathbf{r}$ is the position vector from $O$ to the point located by $x_\alpha$ on the reference surface.

After deformation, the particle that had position vector $\hat{\mathbf{r}}$ in the undeformed state now has the position vector $\hat{\mathbf{R}}$ in the deformed state expressed as

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3\mathbf{B}_3(x_1, x_2) + w_i(x_1, x_2, x_3)\mathbf{B}_i(x_1, x_2) \tag{2}$$

where $\mathbf{B}_i$ forms an orthonormal triad for the deformed configuration and $w_i$ are the warping functions, which are introduced to accommodate all possible deformation other than those described by $\mathbf{R}$ and $\mathbf{B}_i$. Equation (2) can be considered as a change of variable for $\hat{\mathbf{R}}$ in terms of $\mathbf{R}$, $\mathbf{B}_i$, and $w_i$, which is 6 times redundant. The redundancy can be removed by choosing proper definitions for $\mathbf{R}$ and $\mathbf{B}_i$. One can define $\mathbf{R}$ as the average position through the thickness, which implies that the warping functions must satisfy the following three constraints:

$$\langle w_i(x_1, x_2, x_3)\rangle = 0 \tag{3}$$

where the angle brackets denote the definite integral through the thickness of the plate. This implies

$$\mathbf{R}(x_1, x_2) = \langle\hat{\mathbf{R}}(x_1, x_2, x_3)\rangle \neq \hat{\mathbf{R}}(x_1, x_2, 0) \tag{4}$$

It is emphasized that the surface defined by $\mathbf{R}$ may not correspond to the deformed middle surface, but it can be shown to be a smooth surface as long as the plate structure is uniform in the $x_1$–$x_2$ plane.

Another two constraints can be specified by choosing $\mathbf{B}_3$ as a unit vector normal to the reference surface of the deformed plate. It is

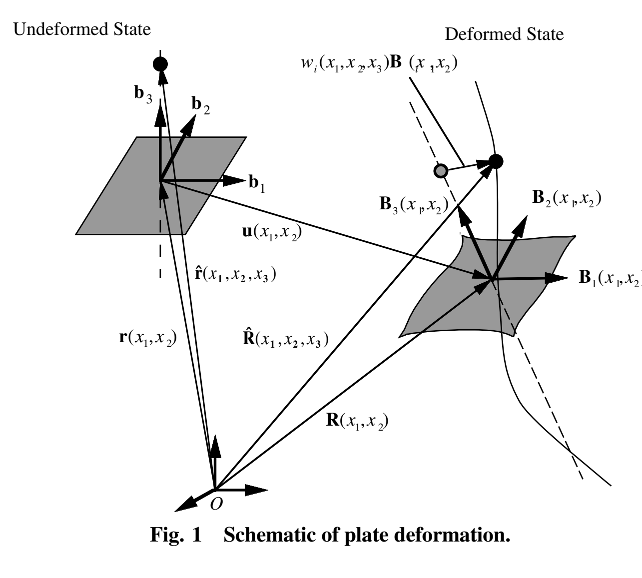


**Fig. 1 Schematic of plate deformation.**

noted that this choice has nothing to do with the famous Kirchhoff hypotheses. In the Kirchhoff hypotheses, no local deformation of the transverse normal is allowed. However, by introducing warping functions, unknown three-dimensional functions to be solved for later, we accommodate all possible deformation. Because $\mathbf{B}_\alpha$ can freely rotate around $\mathbf{B}_3$, we can introduce the last constraint as

$$\mathbf{B}_1 \cdot \mathbf{R}_{,2} = \mathbf{B}_2 \cdot \mathbf{R}_{,1} \tag{5}$$

Based on the concept of decomposition of the rotation tensor [25], the Jauman–Biot–Cauchy strain components for small local rotation are given by

$$\Gamma_{ij} = \tfrac{1}{2}(F_{ij} + F_{ji}) - \delta_{ij} \tag{6}$$

where $\delta_{ij}$ is the Kronecker symbol and $F_{ij}$ are the mixed-basis components of the deformation gradient tensor such that

$$F_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j \tag{7}$$

Here $\mathbf{G}_i = \partial \hat{\mathbf{R}}/\partial x_i$ is the covariant basis vector of the deformed configuration and $\mathbf{g}^k$ is the contravariant base vector of the undeformed configuration and $\mathbf{g}^k = \mathbf{g}_k = \mathbf{b}_k$. One can obtain $\mathbf{G}_k$ with the help of the following definition of two-dimensional generalized strains:

$$\mathbf{R}_{,\alpha} = \mathbf{B}_\alpha + \varepsilon_{\alpha\beta}\mathbf{B}_\beta \tag{8}$$

$$\mathbf{B}_{i,\alpha} = (-K_{\alpha\beta}\mathbf{B}_\beta \times \mathbf{B}_3 + K_{\alpha 3}\mathbf{B}_3) \times \mathbf{B}_i \tag{9}$$

where $\varepsilon_{\alpha\beta}$ and $K_{\alpha\beta}$ are the 2-D generalized strains and $()_{,\alpha} = \partial()/\partial x_\alpha$. For a geometrically nonlinear analysis, we can assume that the strains are small compared with unity and warpings are of the order of strain. Neglecting higher-order terms, one can express the 3-D strain field as

$$\begin{aligned} \Gamma_e &= \varepsilon + x_3\kappa + I_1 w_{\parallel,1} + I_2 w_{\parallel,2} \\ 2\Gamma_s &= w'_\parallel + e_1 w_{3,1} + e_2 w_{3,2} \qquad \Gamma_t = w'_3 \end{aligned} \tag{10}$$

where $()' = \partial()/\partial x_3$, $()_\parallel = \lfloor ()_1 ()_2 \rfloor^T$

$$\Gamma_e = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22} \rfloor^T \qquad 2\Gamma_s = \lfloor 2\Gamma_{13} \quad 2\Gamma_{23} \rfloor^T$$
$$\Gamma_t = \Gamma_{33} \qquad \varepsilon = \lfloor \varepsilon_{11} \quad 2\varepsilon_{12} \quad \varepsilon_{22} \rfloor^T$$
$$\kappa = \lfloor K_{11} \quad K_{12} + K_{21} \quad K_{22} \rfloor^T$$

and

$$I_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad e_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad e_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

The strain energy of the plate per unit area can be written as

$$U = \frac{1}{2}\left\langle \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix}^T \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^T & D_s & D_{st} \\ D_{et}^T & D_{st}^T & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix} \right\rangle \tag{11}$$

where $D_e$, $D_{es}$, $D_{et}$, $D_s$, $D_{st}$, and $D_t$ are the appropriate partition matrices of the original 3-D $6 \times 6$ material matrix. It is noted that the material matrix should be expressed in the global coordinates of the undeformed plate $x_i$. This matrix is fully populated for materials having full anisotropy, which are characterized by as many as 21 constants. As shown in our previous work [3,22–24], the virtual work due to applied loads will not affect the classical model and the warping functions can be solved as the stationary points for the first approximation of the strain energy. According to the variational-asymptotic method (VAM) [2], the first approximation of the strain energy can be obtained by neglecting all the in-plane derivatives in the 3-D strain field in Eq. (10), such that

$$\begin{aligned} 2\Pi_0 = {} & 2\left\langle (\varepsilon + x_3\kappa)^T (D_{es}w'_\parallel + D_{et}w'_3) + w'^T_\parallel D_{st} w'_3 \right\rangle \\ & + \left\langle (\varepsilon + x_3\kappa)^T D_e (\varepsilon + x_3\kappa) + w'^T_\parallel D_s w'_\parallel + w'^T_3 D_t w'_3 \right\rangle \end{aligned} \tag{12}$$

The warping field that minimizes the energy expression of Eq. (12), subject to the constraints of Eq. (3), can be obtained by applying the usual procedure of the calculus of variations with the aid of Lagrange multipliers. The resulting warping is

$$w_\parallel = D_{\parallel 1}\varepsilon + D_{\parallel 2}\kappa \qquad w_3 = D_{\perp 1}\varepsilon + D_{\perp 2}\kappa \tag{13}$$

with

$$\begin{aligned} D'_{\parallel 1} &= -D_s^{-1} D_{es}^{*T} \qquad D'_{\parallel 2} = -x_3 D_s^{-1} D_{es}^{*T} \qquad \langle D_{\parallel\alpha} \rangle = 0 \\ D'_{\perp 1} &= -D_{et}^{*T}/D_t^* \qquad D'_{\perp 2} = -x_3 D_{et}^{*T}/D_t^* \qquad \langle D_{\perp\alpha} \rangle = 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} D_t^* = D_t - D_{st}^T D_s^{-1} D_{st} \qquad D_{et}^* = D_{et} - D_{es} D_s^{-1} D_{st} \\ D_{es}^* = D_{es} - D_{et}^* D_{st}^T / D_t^* \end{aligned} \tag{15}$$

Note that interlamina continuity of $D_{\parallel\alpha}$ and $D_{\perp\alpha}$ must be maintained to produce a continuous displacement field. Substituting Eq. (13) back into Eq. (12), we will obtain an asymptotically correct strain energy for CLT as

$$2\Pi_0 = \varepsilon^T A \varepsilon + 2\varepsilon^T B \kappa + \kappa^T D \kappa \equiv \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix}^T K \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \tag{16}$$

where

$$\begin{aligned} A = \langle D_e^* \rangle \qquad B = \langle x_3 D_e^* \rangle \\ D = \langle x_3^2 D_e^* \rangle \quad \text{with} \quad D_e^* = D_e - D_{es} D_s^{-1} D_{es}^{*T} - D_{et} D_{et}^{*T}/D_t^* \end{aligned} \tag{17}$$

and $K$ is a $6 \times 6$ stiffness matrix formed by plate constants as

$$K = \begin{bmatrix} A_{11} & A_{16} & A_{12} & B_{11} & B_{16} & B_{12} \\ A_{16} & A_{66} & A_{26} & B_{16} & B_{66} & B_{26} \\ A_{12} & A_{26} & A_{22} & B_{12} & B_{26} & B_{22} \\ B_{11} & B_{16} & B_{12} & D_{11} & D_{16} & D_{12} \\ B_{16} & B_{66} & B_{26} & D_{16} & D_{66} & D_{26} \\ B_{12} & B_{26} & B_{22} & D_{12} & D_{26} & D_{22} \end{bmatrix} \tag{18}$$

There are three observations to make here for this asymptotically correct CLT:

1) The normal line of undeformed plate does not remain straight and normal to the deformed plate; rather, it deforms in both the normal and in-plane directions in response to deformation involving $\varepsilon$ and $\kappa$. Even if the material of each layer assumes monoclinic symmetry, the normal line will still deform along the normal direction.

2) The strain energy coincides with that of CLT, which is normally derived based on the conflicting assumptions of the plane stress and nondeformable transverse normals.

3) It can be easily observed that neither the normal strain nor the transverse shear strains vanish. The transverse normal stress and shear stresses can be shown to be zero, which are not assumed a priori but are direct consequences of the model.

It is emphasized that no ad hoc assumptions, such as setting the transverse normal strain equal to zero, were used to obtain this solution.

## Extraction of Equivalent Beam Properties

To extract the equivalent beam properties from the asymptotically correct CLT, we need to construct a one-dimensional (2-D) beam model from the 2-D plate model using VAM, which can be specialized for the present purpose in terms of the dimensional reduction from two dimensions to one dimension for general thin-walled beams presented in [16,26–28].

For a strip beam, let us choose $x_1$ along the beam reference line and $x_2$ along the width. Following Yu et al. [27] and Roy and Yu [28], one can express the 2-D plate strains in terms of beam strains as

$$\varepsilon_{11} = \gamma_{11} - \kappa_3 x_2 \qquad 2\varepsilon_{12} = \frac{\partial v_1}{\partial x_2} \qquad \varepsilon_{22} = \frac{\partial v_2}{\partial x_2} \qquad \kappa_{11} = \kappa_2$$
$$2\kappa_{12} = -2\kappa_1 \qquad \kappa_{22} = \frac{\partial^2 v_3}{\partial x_2^2} \tag{19}$$

where $\gamma_{11}$ is the extensional strain, $\kappa_1$ is the torsional strain, $\kappa_2$ is the bending strain in $x_2$ direction, $\kappa_3$ is the bending strain in $x_3$ direction, and $v_i$ are warping functions to account for the warping of the cross section. To ensure a unique mapping between the plate strains and beam strains, four constraints are necessary for the warping functions $v_i$, viz.,

$$\langle\langle v_i \rangle\rangle = 0 \qquad \left\langle\left\langle \frac{\partial v_3}{\partial x_2} \right\rangle\right\rangle = 0 \tag{20}$$

where the double angle brackets denote integration across the width. It is noted the cross section is a line along the width because the thickness has already been reduced in the previous step. Substituting the 2-D strains in Eqs. (19) into the plate energy, Eq. (16), one can solve for the warping functions subject to the constraints in Eqs. (20). For the convenience of calculation, one can express plate strains in matrix form as

$$\begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = P\epsilon + T\psi \tag{21}$$

with

$$P = \begin{bmatrix} 1 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

$$\epsilon = [\,\gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3\,]^T \qquad \psi = [\,2\varepsilon_{12} \quad \varepsilon_{22} \quad \kappa_{22}\,]^T \tag{23}$$

The only unknown degrees of freedom exist in $\psi$. The constraints in Eq. (20) are only needed to recover $v_i$ and have no effect on the arbitrariness of $\psi$. Substituting Eq. (21) into Eq. (16), we obtain

$$2\Pi_0 = \langle\langle (P\epsilon + T\psi)^T K (P\epsilon + T\psi) \rangle\rangle \tag{24}$$

The minimization problem can be carried out in straightforward manner, yielding

$$\psi = -(T^T K T)^{-1} T^T K P \epsilon \tag{25}$$

which can be used along with the constraints in Eq. (20) to solve for the warping functions $v_i$. Substituting Eq. (25) back in Eq. (24), we can obtain a strain energy for the equivalent beam model as

$$2\Pi_0 = \epsilon^T \langle\langle P^T [K - KT(T^T K T)^{-1} T^T K] P \rangle\rangle \epsilon \equiv \epsilon^T S \epsilon \tag{26}$$

where $S$ is a $4 \times 4$ stiffness matrix for the classical beam model that can be calculated in terms of plate constants as

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{12} & S_{22} & S_{23} & 0 \\ S_{13} & S_{23} & S_{33} & 0 \\ 0 & 0 & 0 & S_{44} \end{bmatrix} \tag{27}$$

with

$$\begin{aligned} S_{11} &= b\left(\bar{A}_{11} - \frac{\bar{B}_{12}^2}{\bar{D}_{22}}\right) \qquad S_{12} = -2b\left(\bar{B}_{16} - \frac{\bar{B}_{12}\bar{D}_{26}}{\bar{D}_{22}}\right) \\ S_{13} &= b\left(\bar{B}_{11} - \frac{\bar{B}_{12}\bar{D}_{12}}{\bar{D}_{22}}\right) \qquad S_{22} = 4b\left(\bar{D}_{66} - \frac{\bar{D}_{26}^2}{\bar{D}_{22}}\right) \\ S_{23} &= -2b\left(\bar{D}_{16} - \frac{\bar{D}_{12}\bar{D}_{26}}{\bar{D}_{22}}\right) \qquad S_{33} = b\left(\bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}}\right) \\ S_{44} &= \frac{b^2}{12} S_{11} \end{aligned} \tag{28}$$

and

$$\begin{aligned} \bar{A}_{11} &= A_{11} + \frac{A_{16}^2 A_{22} - 2A_{12}A_{16}A_{26} + A_{12}^2 A_{66}}{A_{26}^2 - A_{22}A_{66}} \\ \bar{B}_{11} &= B_{11} + \frac{A_{12}A_{66}B_{12} + A_{16}A_{22}B_{16} - A_{26}(A_{16}B_{12} + A_{12}B_{16})}{A_{26}^2 - A_{22}A_{66}} \\ \bar{B}_{12} &= B_{12} + \frac{A_{12}A_{66}B_{22} + A_{16}A_{22}B_{26} - A_{26}(A_{16}B_{22} + A_{12}B_{26})}{A_{26}^2 - A_{22}A_{66}} \\ \bar{B}_{16} &= B_{16} + \frac{A_{12}A_{66}B_{26} + A_{16}A_{22}B_{66} - A_{26}(A_{16}B_{26} + A_{12}B_{66})}{A_{26}^2 - A_{22}A_{66}} \\ \bar{D}_{11} &= D_{11} + \frac{A_{66}B_{12}^2 - 2A_{26}B_{12}B_{16} + A_{22}B_{16}^2}{A_{26}^2 - A_{22}A_{66}} \\ \bar{D}_{12} &= D_{12} + \frac{A_{66}B_{12}B_{22} + A_{22}B_{16}B_{26} - A_{26}(B_{16}B_{22} + B_{12}B_{26})}{A_{26}^2 - A_{22}A_{66}} \\ \bar{D}_{22} &= D_{22} + \frac{A_{66}B_{22}^2 - 2A_{26}B_{22}B_{26} + A_{22}B_{26}^2}{A_{26}^2 - A_{22}A_{66}} \\ \bar{D}_{16} &= D_{16} + \frac{A_{66}B_{12}B_{26} + A_{22}B_{16}B_{66} - A_{26}(B_{16}B_{26} + B_{12}B_{66})}{A_{26}^2 - A_{22}A_{66}} \\ \bar{D}_{26} &= D_{26} + \frac{A_{66}B_{22}B_{26} + A_{22}B_{26}B_{66} - A_{26}\left(B_{26}^2 + B_{22}B_{66}\right)}{A_{26}^2 - A_{22}A_{66}} \\ \bar{D}_{66} &= D_{66} + \frac{A_{66}B_{26}^2 - 2A_{26}B_{26}B_{66} + A_{22}B_{66}^2}{A_{26}^2 - A_{22}A_{66}} \end{aligned} \tag{29}$$

where $b$ is the width of the strip. Here, it is assumed for convenience that the strip is uniform along the width. The closed-form expressions are of tremendous practical value because one can directly use the well-known $A$, $B$, and $D$ matrices of CLT to obtain the classical beam properties for more simplified analyses. For isotropic strips, $S_{11}$ is the axial stiffness, $S_{22}$ is the torsional stiffness, and $S_{33}$ and $S_{44}$ are the bending stiffnesses. For composite strips, $S_{12}$ is extension-twist coupling, $S_{13}$ is extension-bending coupling, and $S_{23}$ is bending-twist coupling. In either case, the in-plane bending is uncoupled from other deformations.

This asymptotic derivation also discloses the necessary assumptions to obtain the equivalent beam properties if one prefers more of an engineering approach. Minimization of Eq. (24) is equivalent to setting the laminate resultants $N_{22}$, $N_{12}$, and $M_{22}$ equal to zero. This means one can still let $N_{22} = N_{12} = M_{22} = 0$ to correct the laminate constitutive relations and then introduce ad hoc kinematic assumptions, such as the cross-sectional plane being rigid, and have it lead to the same beam elastic constants. However, in addition to the awkwardness of these conflicting assumptions, the possibility of recovering the cross-sectional warping will be lost. It is emphasized that, in using such an ad hoc approach, relying on the mathematical justification provided by the asymptotic derivation, one does not gain anything.

## Examples

Let $h$ denote the thickness of the strip. If the strip is made of isotropic material with Young's modulus $E$ and Poisson's ratio $\nu$, the nonzero plate constants can be calculated using Eq. (17) along with Eq. (15) as

$$A_{11} = A_{22} = \frac{Eh}{1-\nu^2} \qquad A_{12} = \nu A_{11} \qquad A_{66} = \frac{Eh}{2(1+\nu)}$$
$$D_{11} = D_{22} = \frac{h^2}{12} A_{11} \qquad D_{66} = \frac{h^2}{12} A_{66} \tag{30}$$

and the corresponding nonzero beam constants can be calculated using Eq. (28) along with Eq. (29) as

$$S_{11} = Ebh \qquad S_{22} = \frac{Eh^3 b}{6(1+\nu)} \qquad S_{33} = \frac{Eh^3 b}{12} \qquad S_{44} = \frac{Ehb^3}{12} \tag{31}$$

The plate constants in Eq. (30) and beam constants in Eq. (31) are the well-established formulas tested in engineering practice of isotropic materials for centuries. However, conflicting ad hoc assumptions, such as the Kirchhoff kinematic hypotheses vs the plane-stress assumption for plates or the Euler–Bernoulli kinematic hypotheses vs the uniaxial stress assumption for beams, are not used in the present derivation. In fact, the present derivation 1) provides rigorous justification for the well-known plate and beam constants, 2) proves the validity of the plane-stress assumption for plates and the uniaxial stress assumption for beams, and 3) proves the *invalidity* of the commonly accepted kinematical hypotheses involved in formulating the classical theories for plates and beams.

When structures are made of composite materials, the traditional approach based on ad hoc assumptions meets some limitations because it is not so intuitive to determine, a priori, all the necessary assumptions to construct predictive models. However, the present approach based on VAM is still valid because we only take advantage of the strip geometry, that is, that the thickness is much smaller than the width and the width is much smaller than the length. Here we will use two composite strips as an example to show how reliable beam properties can be extracted from the well-known $A$, $B$, and $D$ matrices of CLT.

Consider a composite strip which is 2 in. wide and 0.2 in. thick. It is made of a composite material with $E_1 = 25 \times 10^6$ psi, $E_2 = E_3 = 10 \times 10^6$ psi, $G_{12} = G_{13} = 5 \times 10^6$ psi, $G_{23} = 2 \times 10^6$ psi, and $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. If the strip is made of a single layer with 15 deg layup orientation, the nonzero plate constants are

$$A = \begin{bmatrix} 4.78747 & 0.617776 & 0.647436 \\ 0.617776 & 1.13462 & 0.151455 \\ 0.647436 & 0.151455 & 2.12278 \end{bmatrix} \times 10^6 \text{ lb/in.}$$
$$D = \begin{bmatrix} 1.59582 & 0.205925 & 0.215812 \\ 0.205925 & 0.378205 & 0.0504849 \\ 0.215812 & 0.0504849 & 0.707594 \end{bmatrix} \times 10^4 \text{ lb} \cdot \text{in.} \tag{32}$$

and the $B$ matrix vanishes. The nonzero beam constants calculated using Eq. (28) along with Eq. (29) are tabulated in Table 1. To check the accuracy of the beam constants extracted directly from CLT, we compare these results with those from the variational asymptotic beam sectional (VABS) analysis [29], a finite element based, 2-D, cross-sectional analysis. A 2-D finite element mesh should be used as input for VABS to carry out the numerical analysis. VABS is neither restricted to thin-walled geometries nor is it based on CLT. The present model has a fair agreement with VABS as shown in Table 1. The percentage error is calculated as

$$\frac{|\text{PRESENT}| - |\text{VABS}|}{|\text{VABS}|} \times 100$$

**Table 1 Beam constants for a single-layer composite strip**

| | Present | VABS | % error |
|---|---|---|---|
| $s_{11}$ | $8.59859 \times 10^6$ | $8.59859 \times 10^6$ | 0.00 |
| $s_{22}$ | $2.99683 \times 10^4$ | $2.78686 \times 10^4$ | 7.53 |
| $s_{23}$ | $-7.62111 \times 10^3$ | $-7.08714 \times 10^3$ | −7.53 |
| $s_{33}$ | $3.06001 \times 10^4$ | $3.04643 \times 10^4$ | 0.45 |
| $s_{44}$ | $2.8662 \times 10^6$ | $2.8662 \times 10^6$ | 0.00 |

**Table 2 Beam constants for a four-layer composite strip**

| | Present | VABS | % error |
|---|---|---|---|
| $s_{11}$ | $7.21485 \times 10^6$ | $7.15539 \times 10^6$ | 0.83 |
| $s_{12}$ | $1.47018 \times 10^5$ | $1.34280 \times 10^5$ | 9.49 |
| $s_{22}$ | $3.58842 \times 10^4$ | $3.28966 \times 10^4$ | 9.08 |
| $s_{33}$ | $2.20417 \times 10^4$ | $2.20109 \times 10^4$ | 0.14 |
| $s_{44}$ | $2.40495 \times 10^6$ | $2.35044 \times 10^6$ | 2.32 |

As explained in Yu et al. [27], the sizable differences in torsion related terms ($S_{22}$ and $S_{23}$) are due to the thin-walled assumption, which is a necessary condition for CLT to be valid. When $h/b$ becomes smaller, the CLT-based beam properties will asymptotically approach VABS results as shown in Hodges [16].

To deal with more realistic layups, let us consider the strip is made of four layers with the asymmetric layup $[30/30/-30/-30 \text{ deg}]$. The nonzero plate constants can be calculated as

$$\times 10^6 \text{ lb/in:}\ A_{11} = 3.95513 \qquad A_{12} = 0.916667$$
$$A_{22} = 2.41667 \qquad A_{66} = 1.40385 \tag{33}$$

$$\times 10^4 \text{ lb} \cdot \text{in:}\ D_{11} = 1.31838 \qquad D_{12} = 0.305556$$
$$D_{22} = 0.805556 \qquad D_{66} = 0.467949 \tag{34}$$

$$\times 10^4 \text{ lb:}\ B_{16} = -4.49667 \qquad B_{26} = -2.16506 \tag{35}$$

The nonzero beam constants calculated using Eq. (28) along with Eq. (29) are tabulated in Table 2. It can be observed that, as the heterogeneity of the structure increases, the difference between the present analytical approach and VABS becomes larger.

## Conclusions

Based on the variational-asymptotic method, a derivation from the classical lamination theory of asymptotically exact equivalent beam properties is presented for a strip. None of the ad hoc approximations that have appeared in the literature for plate theories or beam theories are used. For example, the cross section is *not* assumed to be rigid in its own plane nor is the value of Poisson's ratio set to zero. Instead, the necessary and sufficient engineering assumptions to obtain the correct section constants and stress recovery relations for a strip beam are also presented. Numerical examples are provided to illustrate that the results agree with those from well-established methods of calculating section constants.

## Acknowledgments

This research was supported in part by the Army Vertical Lift Research Center of Excellence at the Georgia Institute of Technology and its affiliate program through a subcontract at Utah State University. The technical monitor is Michael J. Rutkowski.

## References

[1] Hyer, M. W., *Stress Analysis of Fiber-Reinforced Composite Materials*, McGraw–Hill, New York, 1998.
[2] Berdichevsky, V. L., "Variational-Asymptotic Method of Constructing a Theory of Shells," *Journal of Applied Mathematics and Mechanics*, Vol. 43, No. 4, 1979, pp. 664–687. doi:10.1016/0021-8928(79)90152-7
[3] Yu, W., "Mathematical Construction of a Reissner–Mindlin Plate Theory for Composite Laminates," *International Journal of Solids and Structures*, Vol. 42, No. 26, Dec. 2005, pp. 6680–6699. doi:10.1016/j.ijsolstr.2005.02.049
[4] Vlasov, V. Z., *Thin-Walled Elastic Beams*, National Science Foundation, Washington, D.C., 1961.
[5] Bauld, N. R., Jr., and Tzeng, L.-S., "A Vlasov Theory for Fiber-Reinforced Beams with Thin-Walled Open Cross Sections,"

*International Journal of Solids and Structures*, Vol. 20, No. 3, 1984, pp. 277–297.
doi:10.1016/0020-7683(84)90039-8

[6] Chandra, R., and Chopra, I., "Experimental and Theoretical Analysis of Composite I-Beams with Elastic Coupling," *AIAA Journal*, Vol. 29, No. 12, Dec. 1991, pp. 2197–2206.

[7] Wu, X.-X., and Sun, C. T., "Simplified Theory for Composite Thin-Walled Beams," *AIAA Journal*, Vol. 30, No. 12, Dec. 1992, pp. 2945–2951.

[8] Jung, S. N., Nagaraj, V. T., and Chopra, I., "Refined Structural Model for Thin- and Thick-Walled Composite Rotor Blades," *AIAA Journal*, Vol. 40, No. 1, Jan. 2002, pp. 105–116.

[9] Volovoi, V. V., Hodges, D. H., Berdichevsky, V. L., and Sutyrin, V., "Asymptotic Theory for Static Behavior of Elastic Anisotropic I-Beams," *International Journal of Solids and Structures*, Vol. 36, No. 7, 1999, pp. 1017–1043.
doi:10.1016/S0020-7683(97)00341-7

[10] Volovoi, V. V., Hodges, D. H., Cesnik, C. E. S., and Popescu, B., "Assessment of Beam Modeling Methods for Rotor Blade Applications," *Mathematical and Computer Modelling*, Vol. 33, Nos. 10–11, 2001, pp. 1099–1112.
doi:10.1016/S0895-7177(00)00302-2

[11] Bert, C. W., "Simplified Analysis of Static Shear Factors for Beams of Nonhomogeneous Cross Section," *Journal of Composite Materials*, Vol. 7, No. 4, 1973, pp. 525–529.
doi:10.1177/002199837300700410

[12] Vinson, J. R., and Sierakowski, R. L., *The Behavior of Structures Composed of Composite Materials*, 2nd ed., Kluwer Academic, Norwell, MA, 2002.

[13] Reissner, E., and Tsai, W. T., "Pure Bending, Stretching, and Twisting of Anisotropic Cylindrical Shells," *Journal of Applied Mechanics*, Vol. 39, No. 1, March 1972, pp. 148–154.

[14] Johnson, E. R., Vasiliev, V. V., and Vasiliev, D. V., "Anisotropic Thin-Walled Beams with Closed Cross-Sectional Contours," *AIAA Journal*, Vol. 39, No. 12, Dec. 2001, pp. 2389–2393.

[15] Gjelsvik, A., *The Theory of Thin-Walled Bars*, Wiley, New York, 1981.

[16] Hodges, D. H., *Nonlinear Composite Beam Theory*, AIAA, Reston, Virginia, 2006.

[17] Barbero, E. J., Lopez-Anido, R., and Davalos, J. F., "One the Mechanics of Thin-Walled Laminated Composite Beams," *Journal of Composite Materials*, Vol. 27, No. 8, 1993, pp. 806–829.
doi:10.1177/002199839302700804

[18] Whitney, J. M., Browning, C. E., and Mair, A., "Analysis of the Flexure Test for Laminated Composite Materials," *Composite Materials: Testing and Design (Third Conference)*, STP 546, American Society for Testing and Materials, West Conshohocken, PA, 1974, pp. 30–45.

[19] Reddy, J. N., *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, CRC Press, Boca Raton, FL, 2004.

[20] Lopez-Anido, R., Davalos, J. F., and Barbero, E. J., "Experimental Evaluation of Stiffness of Laminated Composite Beam Elements Under Flexure," *Journal of Reinforced Plastics and Composites*, Vol. 14, No. 4, April 1995, pp. 349–361.

[21] Hodges, D. H., Harursampath, D., Volovoi, V. V., and Cesnik, C. E. S., "Non-Classical Effects in Non-Linear Analysis of Pretwisted Anisotropic Strips," *International Journal of Non-Linear Mechanics*, Vol. 34, No. 2, March 1999, pp. 259–277.
doi:10.1016/S0020-7462(98)00023-7

[22] Yu, W., Hodges, D. H., and Volovoi, V. V., "Asymptotic Construction of Reissner-Like Models for Composite Plates with Accurate Strain Recovery," *International Journal of Solids and Structures*, Vol. 39, No. 20, 2002, pp. 5185–5203.
doi:10.1016/S0020-7683(02)00410-9

[23] Yu, W., Hodges, D. H., and Volovoi, V. V., "Asymptotically Accurate 3-D Recovery from Reissner-Like Composite Plate Finite Elements," *Computers and Structures*, Vol. 81, No. 7, 2003, pp. 439–454.
doi:10.1016/S0045-7949(03)00011-7

[24] Yu, W., and Hodges, D. H., "An Asymptotic Approach for Thermoelastic Analysis of Laminated Composite Plates," *Journal of Engineering Mechanics*, Vol. 130, No. 5, 2004, pp. 531–540.
doi:10.1061/(ASCE)0733-9399(2004)130:5(531)

[25] Danielson, D. A., and Hodges, D. H., "Nonlinear Beam Kinematics by Decomposition of the Rotation Tensor," *Journal of Applied Mechanics*, Vol. 54, No. 2, 1987, pp. 258–262.

[26] Volovoi, V. V., and Hodges, D. H., "Theory of Anisotropic Thin-Walled Beams," *Journal of Applied Mechanics*, Vol. 67, No. 3, Sept. 2000, pp. 453–459.
doi:10.1115/1.1312806

[27] Yu, W., Liao, L., Hodges, D. H., and Volovoi, V. V., "Theory of Initially Twisted, Composite, Thin-Walled Beams," *Thin-Walled Structures*, Vol. 43, No. 8, 2005, pp. 1296–1311.
doi:10.1016/j.tws.2005.02.001

[28] Roy, S., and Yu, W., "An Asymptotically Correct Model for Initially Curved and Twisted Thin-Walled Composite Beams," *International Journal of Solids and Structures*, Vol. 44, Nos. 11–12, June 2007, pp. 4039–4052.
doi:10.1016/j.ijsolstr.2006.11.004

[29] Yu, W., Hodges, D. H., Volovoi, V. V., and Cesnik, C. E. S., "On Timoshenko-Like Modeling of Initially Curved and Twisted Composite Beams," *International Journal of Solids and Structures*, Vol. 39, No. 19, 2002, pp. 5101–5121.
doi:10.1016/S0020-7683(02)00399-2

F. Pai
*Associate Editor*